JOURNAL OF COMPUTER AND SYSTEM SCIENCES 9, 48–55 (1974)

Difunctionally Induced State Machines

EUGENE M. NORRIS

University of South Carolina, Columbia, South Carolina 29208

Received April 11, 1973

1. INTRODUCTION

The following is a very slightly modified restatement of Theorem 3 in Bednarek and Wallace's paper [3].

THEOREM 1 (Bednarek–Wallace). *Let T and X be both compact or both discrete spaces, with T a topological groupoid, and suppose E and F are closed equivalence relations on T and on X, respectively, with E in addition such a congruence that T/E is a semigroup. If R is a closed relation from $T \times X$ into X satisfying, for all t, t' in T and all x in X, the conditions*

(1) $T \times X = RX$,

(2) $R^{-1} \circ (E \times F) \circ R \subseteq F$,

(3) $(tt', x)R \times (t, z)R \subseteq F$ *for all z in* $(t', x)R$,

then T/E acts on X/F uniquely to make the following diagram of projections and quotient maps commutative.

$$\begin{array}{ccc} T \times X & \longleftarrow R \longrightarrow & X \\ \downarrow & & \downarrow \\ T/E \times X/F & \longrightarrow & X/F \end{array}$$

Indeed, the theorem as stated in [3] hypothesizes that T is a semigroup, but the proof uses only that T/E is a semigroup. As Bednarek and Wallace noted, condition (3) is needed only to insure that the induced function is an action; in fact, this theorem is a corollary to their general Induced Function Theorem.

There is a strong converse to Theorem 1 (our Theorem 2 below), which is interesting for a couple of reasons; roughly, when the quotient action exists for a given E and F, there is always a *difunctional* relation R satisfying conditions (1)–(3) of Theorem 1, and, hence, one may always assume from the outset that R is difunctional. Theorem 1 includes various theorems on inducing quotient actions from a given action, as the authors of [3] note. Since there is some interest in nondeterministic automata with a transition monoid consisting of relations of a type more general than functions, we are thus led to study "relational automata" at a rather high level of generality. If we assume that the transition relation in a machine is difunctional, several interesting things evolve. For example, the collection of "state transition" relations form a semigroup of difunctional relations, which under appropriate topological hypotheses becomes a compact Hausdorff space. Since the composition of difunctional relations is not in general difunctional, this result has some significance and ought to be exploited. We initiate therefore in this paper a study of generalized semigroup actions which we call *actoids*. The principal result of this paper is to canonically produce from a given difunctional relation R a pair of equivalence relations E and F so that the resulting quotient action is "nice" from a machine-theoretic point of view. This construction is the subject of Sections 3 and 4 of this paper. In Section 2 we establish notation and state some needed results from the folklore of relation theory. An exposition of relation theory may be found in the paper by Bednarek, Magill, and Norris contained in the forthcoming volume [6] edited by Preston Hammer.

2. Relations

(i) *Algebra*

If R is a relation from A to B and S is a relation from B to C, their *composition* is the relation $R \circ S = \{(a, c)\colon (a, b) \in R \text{ and } (b, c) \in S \text{ for some } b\}$. The *inverse* of R is the relation $R^{-1} = \{(b, a)\colon (a, b) \in R\}$. If R is a relation from A to B and if C and D are subsets of A and B, respectively, then we define the *slice* $CR = \{b \in B\colon (c, b) \in R$ for some $c \in C\}$ and dually define $RD = DR^{-1}$. The term "slice" (French, *tranche*) is due to Riguet [13, 14], if not to an earlier writer. Since CR is the projection into B of the set $R \cap (C \times B)$, the terminology is not inappropriate.

We write aR rather than $\{a\}R$ and so forth for simplicity's sake. Riguet [13, 14] has called a relation R from A to B *difunctional* if $R \circ R^{-1} \circ R \subseteq R$. The class of difunctional relations includes all partially defined functions from A into B and, when $A = B$, all symmetric, transitive relations. We shall use Riguet's results freely in the sequel, and we note in particular that if R is a difunctional relation from A to B then $R \circ R^{-1}$ and $R^{-1} \circ R$ are equivalence relations on RB and AR, respectively. In fact, $RB/R \circ R^{-1} = \{Rb\colon b \in B\}$ and $AR/R^{-1} \circ R = \{aR\colon a \in A\}$. We observe the crucial property here of difunctionality; namely, that two slices Rb, Rb' (respectively, aR, $a'R$) are either disjoint or coincide. This property characterizes difunctionality. Difunctional relations have been discussed in several recent papers [2, 7, 8, 12]. By way of orientation, if R were a function, what we call aR (respectively, Rb) would be just the set containing the single point $R(a)(R^{-1}(b))$. It then evolves that for two functions R, S, the point in $a[R \circ S]$ is just $S(R(a))$.

(ii) *Topology*

On the topological side we shall require the following facts, all of which are known to be true. If A, B, and C are all compact Hausdorff spaces and R and S are closed relations, then $R \circ S$ is closed ("closed" means with respect to the product topology on $A \times B$, etc.) If A is compact, then A/R is compact Hausdorff. If A and B are compact, R is a closed relation and C and D are closed subsets of A and B, respectively, then CR and RD are closed. This property, that RD is closed whenever D is closed, is sometimes called *upper semicontinuity* [9], and it has the following consequence. If $aR \subseteq U$ and U is open, then there is an open neighborhood V of a with the property that $VR \subseteq U$. Finally, if $R \subseteq A \times B$ and $S \subseteq C \times D$, we put $R \times S = \{((t, x), (t', x')): (t, t') \in R \text{ and } (x, x') \in S\}$. If E and F are closed equivalences on compact Hausdorff spaces A and B, then $E \times F$ is a closed equivalence relation on $A \times B$, and $A \times B/E \times F$ is homeomorphic to $A/E \times B/F$.

3. Relational Machines

The following converse of Theorem 1 is true.

Theorem 2 (Difunctional Sufficiency Theorem). *Let T and X be both compact Hausdorff or both discrete spaces, with T a topological groupoid, and suppose that E and F are closed equivalence relations on T and X respectively, with E such a congruence on T that T/E is a semigroup. If T/E acts on X/F, then there is a closed difunctional relation R from $T \times X$ into X which satisfies conditions* (1)–(3) *of Theorem* 1 *and for which the diagram of Theorem* 1 *is commutative.*

Proof. Let R be defined by the condition $((t, x), y) \in R$ if and only if $\bar{t}\bar{x} = \bar{y}$, where in general $\bar{z}$ denotes an equivalence class containing the element z. The surjectivity of the quotient maps $T \to T/E$ and $X \to X/F$ guarantees that $T \times X = RX$. To see that (2) holds, suppose that $(x, x') \in R^{-1} \circ (E \times F) \circ R$, so that there are elements t, t' in T, y, y' in X for which $((t, y), x) \in R$, $((t, y), (t', y')) \in E \times F$ and $((t', y'), x') \in R$. Then, since $(t, t') \in E$ and $(y, y') \in F$, we see that $\bar{x} = \bar{t}\bar{y} = \bar{t}'\bar{y}' = \bar{x}'$, i.e., $(x, x') \in F$. Condition (3) also holds, for if $z \in (t', x)R$, let $y \in (tt', x)R$ and $y' \in (t, z)R$ and compute: $\bar{y} = (\overline{tt'})\bar{x}$, while $\bar{y}' = \bar{t}\bar{z} = \bar{t}(\bar{t}'\bar{x}) = (\bar{t}\bar{t}')\bar{x}$. Since E is a congruence, $\overline{tt'} = \bar{t}\bar{t}'$ so that $\bar{y} = \bar{y}'$, proving (3). To see that R is closed, one need only observe that it is the inverse image of the diagonal of the Hausdorff space X/F under the continuous map $[m \circ (a \times b)] \times b$ fashioned from the groupoid multiplication m and canonical quotient maps a: $T \to T/E$ and b: $X \to X/F$. R is immediately seen to be difunctional.

In connection with the difunctionality of R in Theorem 2, and in view of the fact that not much is known about difunctionality in general, the following easily proved result is of interest. It will not be used in this paper.

PROPOSITION 3. *If $f, g\colon A \to B$ are any two functions and E is any difunctional relation from B to B, then $(f \times g)^{-1}(E)$ is difunctional.*

We are interested in the rest of this paper in structures of the form (T, X, R), where T and X will denote compact Hausdorff spaces and R a closed difunctional relation from $T \times X$ into X which satisfies $T \times X = RX$. Of course, if T and X are finite sets they are compact Hausdorff in the discrete topology, so our results have meaning for finite structures. A study of a quite general class of relational automata appears in [4]. The special case in which R is a function (necessarily continuous) has been studied before as a *topological machine* or *act* [1, 2, 10, 11]. We will be guided by that special case in our choice of terminology. In particular, choosing a point t in T (respectively, a point x in X) induces a relation ϕ_t from X into X (respectively, a relation Ψ_x from T into X) called a *left* (*right*) *translation* of (T, X, R). These relations are defined as follows.

$$\phi_t = \{(x, y)\colon ((t, x), y) \in R\},$$
$$\Psi_x = \{(t, y)\colon ((t, x), y) \in R\}.$$

Loosely speaking, ϕ is just the restriction of R to $(\{t\} \times X) \times X)$ and Ψ_x is just the restriction of R to $(T \times \{x\}) \times X$.

PROPOSITION 4. *If R is a closed difunctional relation from $T \times X$ into X for which $T \times X = RX$ and if T and X are Hausdorff spaces, then each ϕ_t and each ψ_t is a closed difunctional relation; furthermore, $X = \phi_t X$ and $T = \psi_x X$ for each $t \in T$ and each $x \in X$.*

Proof. If t is in T then $\phi_t = p_{2,3}((\{t\} \times X) \times X \cap R)$ is the continuous image of a compact set, where $p_{2,3}$ is the function defined by $p_{2,3}((t, x), y) = (x, y)$; hence ϕ_t is a closed subset of $X \times X$. To see that $\phi_t X = X$ we need only show that $X \subseteq \phi_t X$; if $x \in X$ then $T \times X = RX$ implies that for some y, $((t, x), y) \in R$, so that $x \in \phi_t X$. In a similar way one sees that $T = \psi_x X$. To see that ϕ_t is difunctional, suppose that (x, y), (x', y), and (x', y') all belong to ϕ_t; then $((t, x), y)$, $((t, x'), y)$, and $((t, x'), y')$ all belong to the difunctional relation R, so that $((t, x), y')$ also belongs to R, which in turn implies that $(x, y') \in \phi_t$, so that ϕ_t is difunctional. In a similar fashion one sees that ψ_x is also difunctional.

We digress briefly to mention an interesting consequence of the last proposition. If, given spaces A and B, we write $D(A, B)$ (respectively, $F(A, B)$) for the set of all closed difunctional relations from A to B for which $A = RB$ (respectively, the set of all functions from A to B), then we have shown that there are injections Φ and Ψ mapping $D(T \times X, X)$ into $F(X, D(T, X))$ and $F(T, D(X, X))$, respectively. For convenience we have written ϕ_t, Ψ_x for $[\Phi(R)](t)$, $[\Phi(R)](x)$, as we shall consider in the sequel only a fixed relation R in $D(T \times X, X)$. In general Φ and Ψ are not bijective and hence we do not have an "exponential law" but just an exponential inequality, which, fortunately, is sufficient for our purposes.

We proceed now to construct a pair of equivalence relations on T and X, given a relation R from $T \times X$ into X. Let

$$E = \{(t, t'): \phi_t = \phi_{t'}\}$$

and

$$F = \{(x, x'): \psi_x = \psi_{x'}\}.$$

It is immediate that E and F so defined are equivalences.

Proposition 5. *Under the standing hypotheses on T, X, and R, the relations E and F are closed.*

Proof. We only show that E is closed, the proof for F being entirely similar. If $(t, t') \notin E$, then $\phi_t \neq \phi_{t'}$, so that there exists a point (x, y) in one of them and not in the other. It is surely no loss of generality to suppose that $(x, y) \in \phi_t \backslash \phi_{t'}$, i.e., $y \in x\phi_t \backslash x\phi_{t'}$, or, equivalently, $y \in t\psi_x \backslash t'\psi_x$. Since ψ_x is difunctional as a result of Proposition 4, we may conclude that $t\psi_x \cap t'\psi_x = \varnothing$, and hence, since ψ_x is closed and X is normal, there exist disjoint open sets U_0 and V_0 with $t\psi_x \subseteq U_0$ and $t'\Psi_x \subseteq V_0$. Since ψ_x is upper semicontinuous, there are open sets U and V containing t and t', respectively, so that $U\psi_x \subseteq U_0$ and $V\psi_x \subseteq V_0$. It is easy to see that $(U \times V) \cap E = \varnothing$, for if (s, s') is in $U \times V$ and (x, z) is a point in ϕ_s then $z \in s\psi_x \subseteq U_0 \subseteq X \backslash V_0$, which implies $(x, z) \notin \phi_{s'}$, i.e., $\phi_s \neq \phi_{s'}$. From this we conclude that E is closed.

The hypothesis that R is difunctional is needed here to insure that ψ_x is difunctional. E and F may fail to be closed if R is not difunctional.

4. An Induced Action

If $(T, X, \cdot)$ is an act, then for all s, t in T and all x in X it is the case that $s \cdot (t \cdot x) = (st) \cdot x$, or, equivalently, $\phi_t \circ \phi_s = \phi_{st}$ in our notation. This latter notation is convenient for generalization. We shall call (T, X, R) an *actoid* if T is a topological groupoid (that is to say T is equipped with a continuous binary operation), X is a space and R is a closed difunctional relation from $T \times X$ into X satisfying $T \times X = RX$ and for which $\phi_t \circ \phi_s = \phi_{st}$ for all s and t in T. An actoid (T, X, R) will be called compact Hausdorff if both T and X are compact Hausdorff spaces.

We recall from [10, 11] that if a semigroup A acts on a space B so that for each distinct pair x, y in B there is some s in A so that $sx \neq sy$ (respectively, for each s, t in A there is some x in B so that $sx \neq tx$), then the action is called *left effective* (*right effective*).

THEOREM 6. *Suppose that (T, X, R) is a compact Hausdorff actoid. Then there exist such closed equivalence relations E and F on T and X such that T/E acts both left and right effectively on X/F, and the diagram below, of projections and quotient maps, commutes.*

$$\begin{array}{ccccc} T \times X & \longleftarrow & R & \longrightarrow & X \\ \downarrow & & & & \downarrow \\ T/E \times X/F & & \longrightarrow & & X/F \end{array}$$

We defer the proof of this theorem until after we have established two lemmas which are needed in computation. An inspection of the proof of Theorem 6 indicates that the metatheorem, Theorem 1 implies Theorem 6, is true. The status of the converse of this metatheorem is not known.

LEMMA 7. *If t and t' are in T then the condition $\phi_{t'} \circ \phi_t = \phi_{tt'}$ is equivalent to the condition $(tt', x)R = (t, y)R$ for all x in X and all y in $(t', x)R$.*

Proof. Fix a point y in $(t', x)R$, so that $(x, y) \in \phi_{t'}$. If z is in $(t, y)R$ then (y, z) is in ϕ_t, so that $(x, z) \in \phi_{t'} \circ \phi_t = \phi_{tt'}$, i.e., $z \in (tt', x)R$, proving that $(t, y)R \subseteq (tt', x)R$; the difunctionality of R implies then that the two sets are equal. To see the converse implication, suppose that $(x, z) \in \phi_{t'} \circ \phi_t$ so that for some y, $(x, y) \in \phi_{t'}$ and $(y, z) \in \phi_t$, and hence $y \in (t', x)R$. Then $z \in (t, y)R = (tt', x)R$ implies that $(x, z) \in \phi_{tt'}$, so that $\phi_{t'} \circ \phi_t \subseteq \phi_{tt'}$. If we let $(x, z) \in \phi_{tt'}$ then there is some y in $(t', x)R$, since $T \times X = RX$. Since $z \in (tt', x)R = (t, y)R$ we have $(x, y) \in \phi_{t'}$ and $(y, z) \in \phi_t$, i.e., $(x, z) \in \phi_{t'} \circ \phi_t$ so that the two relations are equal.

LEMMA 8. *If s, t and t' are in T and x is in X, then the condition $\phi_{t'} \circ \phi_t = \phi_{tt'}$ implies that $(s(tt'), x)R = ((st)t', x)R$.*

Proof. The following assertions are equivalent. $y \in (s(tt'), x)R$, $(x, y) \in \phi_{s(tt')} = \phi_{tt'} \circ \phi_s = (\phi_{t'} \circ \phi_t) \circ \phi_s = \phi_{t'} \circ (\phi_t \circ \phi_s)$, $y \in ((st)t', x)R$.

Proof of Theorem 6. We will see that the hypotheses of Theorem 1 are satisfied, using the relations E and F defined in Section 3. To begin, our condition $\phi_{t'} \circ \phi_t = \phi_{tt'}$ implies that E is a congruence, for if (s, s') and (t, t') are in E then $\phi_{st} = \phi_t \circ \phi_s = \phi_{t'} \circ \phi_{s'} = \phi_{s't'}$ so that $(st, s't')$ is in E. Furthermore as is immediately verified, $(s(tu), (st)u)$ is in E for each s, t and u in T, so that T/E is a semigroup. To see that

$R^{-1} \circ (E \times F) \circ R \subseteq F$, suppose that $((t, z), x) \in R$, $((t, z), (t', z')) \in E \times F$ and $((t', z'), x') \in R$; we wish to show that $\psi_x = \psi_{x'}$. To this end, first observe that each of the following assertions is equivalent to $((t, z), x)$ being in R. $x \in (t, z)R$, $(t, x) \in \psi_2 = \psi_{2'}$, $x \in (t, z')R$, $(z', x) \in \phi_t = \phi_{t'}$, $((t', z'), x) \in R$, $x \in (t', z')R$. Since x is thus in both $(t, z)R$ and $(t', z')R$, the difunctionality of R implies that $(t, z)R = (t', z')R$. Then, since $x \in (t, z)R$ and $x' \in (t', z')R$, the following assertions are equivalent. $(s, y) \in \psi_x$, $y \in (s, x)R = (st, z)R$, $(z, y) \in \phi_{st} = \phi_t \circ \phi_s = \phi_{t'} \circ \phi_s$, $(st', y) \in \psi_z = \psi_{z'}$, $y \in (st', z')R = (s, x')R$, $(s, y) \in \psi_{x'}$; hence $\psi_x = \psi_{x'}$.

To verify hypothesis (3) of Theorem 1, fix z in $(t', x)R$ and let $y \in (tt', x)R$ and $y' \in (t, z)R$. If s is any point in T then Lemma 8 and these conditions allow one to see that $(s, y)R = (s(tt'), x)R = ((st)t', x)R = (st, z)R = (s, y)R$, or equivalently, $\psi_y = \psi_{y'}$. Since we have satisfied the hypotheses of Theorem 1, the action of T/E on X/F exists as asserted. The left and right effectivity of the action is a consequence of the commutativity of the diagram.

In case (T, X, R) is a compact topological transformation group (we write the group on the left, contrary to common practice), the collection $\{\phi_t : t \in T\}$ is a subsemigroup [5]. In the more general case that (T, X, R) is an action with X locally compact Hausdorff, the collection $\{\phi_t : t \in T\}$ is a subsemigroup of X^X, which is known to be a topological semigroup. It is of interest to find a "nice" topological semigroup of relations which contains $\{\phi_t : t \in T\}$, for the closure of this set is the natural candidate for the title of enveloping semigroup. The thought immediately comes to mind that, since each ϕ_t is a closed subset of $X \times X$, an appropriate topologization is the Michael topology on $2^{X \times X}$. Unfortunately, simple examples show that even on a compact Hausdorff space, composition of closed relations need not be a continuous operation. In the general case of an actoid, each ϕ_t is in the set $D(X, X)$ defined above; but since the composition of difunctional relations may fail to be difunctional, $D(X, X)$ may fail to be even a groupoid, and therefore the problem seems quite difficult. Of course, under the hypothesis that (T, X, R) is a compact Hausdorff actoid we see from Proposition 5 that E is a closed equivalence relation such that T/E is a compact semigroup. Since the points of T/E are in one-to-one correspondence with $\{\phi_t : t \in T\}$ (in fact by a semigroup isomorphism), a compact Hausdorff topology is inherited by $\{\phi_t : t \in T\}$, making it into a compact semigroup. Without compactness, not much can be said about the enveloping semigroup in general.

References

1. A. R. Bednarek and E. M. Norris, Congruences and ideals in machines, *Rev. Roumaine Math. Pures Appl.* **15** (1970), 193–199.
2. A. R. Bednarek and E. M. Norris, Induced function theorems in topology, *Math. Systems Theory* **8** (1974), to appear.

3. A. R. Bednarek and A. D. Wallace, A relation-theoretic result with applications in topological algebra, *Math. Systems Theory* **1** (1967), 217–224.
4. W. Brauer, "Zu den Grundlagen einer Theorie topologischer sequentieller Systeme und Automaten," Berichte der Gesellschaft für Mathematik und Datenverarbeitung, No. 31, Bonn, Germany, 1970.
5. Robert Ellis, "Lectures on Topological Dynamics," Benjamin, New York, 1969.
6. Preston Hammer (Ed.), "Advances in Mathematical Systems Theory," Vol. II, Pennsylvania State University Press (forthcoming).
7. J. Lambek, Goursat's theorem and the Zassenhaus lemma, *Canad. J. Math.* **10** (1958), 45–56.
8. S. MacLane, An algebra of additive relations, *Proc. Nat. Acad. Sci.* **47** (1961), 1043–1051.
9. E. Michael, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.* **71** (1951), 152–182.
10. E. M. Norris, Some Structure Theorems for Topological Machines, Dissertation, University of Florida, 1969.
11. E. M. Norris, Inverse limits and embedding of compact totally disconnected machines, *Math. Systems Theory* **5** (1971), 89–94.
12. E. M. Norris, Relationally induced semigroups, *Pacific J. Math.* **48** (1973), 203–207.
13. J. Ruguet, Relations binaires, fermetures, correspondances de Galois, *Bull. Soc. Math. France* **76** (1948), 114–155.
14. J. Ruguet, Quelques propriétés des relations difonctionelles, *C. R. Acad. Sci. Paris* **230** (1950), 1999–2000.